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THE CONVEX FLOATING BODY AND POLYHEDRAL APPROXIMATION

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ABSTRACT

We consider the convex floating body of a polytope and polyhedral approximation of a convex body.

In [Schü-W] we found for the convex floating body K_δ of a convex body K

$$\lim_{\delta\to 0} c_n \frac{\mathrm{vol}_n(K) - \mathrm{vol}_n(K_\delta)}{\delta^{2/(n+1)}} = \int_{\partial K} \kappa(x)^{1/(n+1)} d\mu(x)$$

where $\kappa(x)$ is the generalized Gauss-Kronecker curvature. In particular, for polytopes these expressions equal zero. It follows from [B-L] that the order of magnitude of $\mathrm{vol}_n(P) - \mathrm{vol}_n(P_\delta)$ for a polytope P is $\delta(\ln(1/\delta))^{n-1}$. We give here a precise formula. It turns out that we get the same expression for P and its polar P^*. We apply this formula to estimate the symmetric distance between a polytope and a convex body. The main difference to known estimates [Grub], [Schn_1], [Schn_2] is that we do not assume that ∂C is C^2.

0. Notation

Hyperplanes are usually denoted by H. The closed halfspaces associated with H are denoted by H^- and H^+. The polar of a convex body K is given by K^*. The convex hull of sets $M_1, \dots, M_m$ is denoted by

$$[M_1, \dots, M_m].$$

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1. The convex floating body of a polytope

Let P be a convex polytope with nonempty interior in $\mathbf{R}^n$. The set of its k-dimensional faces is denoted by $\mathrm{fac}_k(P)$. The extreme points are also denoted by $\mathrm{ext}(P)$. We define

$$\phi_1(P) = 2 \qquad \text{if } P \subseteq \mathbf{R}.$$

If $P \subseteq \mathbf{R}^n$, $n \geq 2$, then we choose for every extreme point $x \in \mathrm{ext}(P)$ a hyperplane H_x that separates x from all other extreme points, i.e., $x \in \mathring{H}_x^-$ and all other extreme points are in $\mathring{H}_x^+$. We put

$$\phi_n(P) = \sum_{x \in \mathrm{ext}(P)} \phi_{n-1}(P \cap H_x). \tag{1.1}$$

We define

$$\psi_1(P) = 2 \qquad \text{if } P \subseteq \mathbf{R},$$

$$\psi_n(P) = \sum_{F \in \mathrm{fac}_{n-1}(P)} \psi_{n-1}(F) \qquad \text{if } P \subseteq \mathbf{R}^n, \quad n \geq 2. \tag{1.2}$$

We have, in particular, that $\phi_2(P) = \psi_2(P) = 2\#\mathrm{ext}(P)$. For an n-dimensional simplex S we get $\phi_n(S) = \psi_n(S) = (n+1)!$.

For the unit balls of l_n^1 and l_n^∞ we get $\psi_n(B_1^n) = \psi_n(B_\infty^n) = 2^n n!$.

Lemma 1.1. *Let P be a convex polytope with nonempty interior in $\mathbf{R}^n$. Then we have for all $n \in \mathbf{N}$*

(i) $\phi_n(P) = \psi_n(P)$,

(ii) *$\psi_n(P)$ equals the cardinality of the set of all sequences $(f_0, f_1, \dots, f_{n-1})$ where $f_i \in \mathrm{fac}_i(P)$, $i = 0, \dots, n-1$, and $f_0 \subset f_1 \subset \cdots \subset f_{n-1}$.*

(iii) *Suppose that $0 \in \mathring{P}$. Then $\psi_n(P) = \psi_n(P^*)$.*

Proof. We clearly have the equality for $n = 1$ and $n = 2$. Suppose now that the assertion is true for all integers between 1 and $n-1$:

$$\phi_n(P) = \sum_{x \in \mathrm{ext}(P)} \phi_{n-1}(P \cap H_x) = \sum_{x \in \mathrm{ext}(P)} \psi_{n-1}(P \cap H_x).$$

We have by definition of ψ_n

$$\psi_{n-1}(P \cap H_x) = \sum_{F \ni x} \psi_{n-2}(F \cap H_x)$$

because all $n-2$ dimensional faces of $P \cap H_x$ are of the form $P \cap H_x \cap F$ where F is an $n-1$ dimensional face of P.

Therefore we get

$$\phi_n(P) = \sum_{x\in \mathrm{ext}(P)} \sum_{F\ni x} \psi_{n-2}(F\cap H_x) = \sum_{F\in \mathrm{fac}_{n-1}(P)} \sum_{x\in F} \psi_{n-2}(F\cap H_x).$$

By our assumption we get

$$\phi_n(P) = \sum_{F\in \mathrm{fac}_{n-1}(P)} \sum_{x\in F} \phi_{n-2}(F\cap H_x).$$

And by the definition of ϕ_n

$$\phi_n(P) = \sum_{F\in \mathrm{fac}_{n-1}(P)} \phi_{n-1}(F).$$

Again by our assumption

$$\phi_n(P) = \sum_{F\in \mathrm{fac}_{n-1}(P)} \psi_{n-1}(F) = \psi_n(F).$$

(iii) is a consequence of (ii) and [G, section 3.4]. ■

THEOREM 1.2. *Let P be a convex polytope with nonempty interior in $\mathbf{R}^n$. Then we have*

$$\lim_{\delta\to 0} \frac{\mathrm{vol}_n(P) - \mathrm{vol}_n(P_\delta)}{\delta\left(\ln \frac{1}{\delta}\right)^{n-1}} = \frac{1}{n!}\,\frac{1}{n^{n-1}}\,\phi_n(P).$$

LEMMA 1.3. *Let S be a simplex with nonempty interior in $\mathbf{R}^n$ spanned by $z_0, z_1, \ldots, z_n$, and $\delta \le \frac{1}{2}\mathrm{vol}_n(S)$.*

(i) *Let H_δ denote hyperplanes that cut off a set of volume δ from S and so that $z_0 \in \mathring{H}_\delta^-$ but none of the other extreme points. Then we have*

$$\begin{aligned}\frac{1}{n^{n-1}}\,\delta\left(\ln \frac{\mathrm{vol}_n(S)}{\delta}\right)^{n-1} &\le \mathrm{vol}_n\left(S\cap\left(\bigcup H_\delta^-\right)\right)\\ &\le \frac{1}{n^{n-1}}\,\delta\left(\ln\frac{n^n\mathrm{vol}_n(S)}{\delta}\right)^{n-1}\\ &\quad + n\delta\left(\ln\left(\frac{\mathrm{vol}_n(S)n!}{\delta}\right)\right)^{n-2}.\end{aligned}$$

(ii) *Let H_δ denote hyperplanes that cut off a set of volume δ from S so that at least z_0, z_1 are in $\mathring{H}_\delta^-$. Then we have*

$$\mathrm{vol}_n\left(S\cap\left(\bigcup H_\delta^-\right)\right) \le c_n\delta\left(\ln\frac{\mathrm{vol}_n(S)}{\delta}\right)^{n-2}.$$

(iii) *Let H_δ denote hyperplanes that cut off a set of volume δ from S. Then we have*

$$\mathrm{vol}_n\left(S\cap\left(\bigcup H_\delta^-\right)\right)\le\frac{\delta(n+1)}{n^{n-1}}\left(\ln\frac{n^n\mathrm{vol}_n(S)}{\delta}\right)^{n-1}+c_n\delta\left(\ln\frac{\mathrm{vol}_n(S)}{\delta}\right)^{n-2}.$$

PROOF. (i) We may assume that S is spanned by 0, e_1, e_1+e_2, $e_1+e_3,\dots,e_1+e_n$. Let $t_1e_1\in H_\delta$ and $t_i(e_1+e_i)\in H_\delta$ for $i=2,\dots,n$. It follows from the assumptions that $0\le t_i\le 1$, $i=1,\dots,n$ and

$$\delta=\mathrm{vol}_n(S\cap H_\delta^-)=\frac{1}{n!}\prod_{i=1}^n t_i. \tag{1.3}$$

We show that H_δ touches the surface

$$x_1=\sum_{i=2}^n x_i+\frac{\delta n!}{n^n}\left(\prod_{i=2}^n x_i\right)^{-1} \tag{1.4}$$

at $x=t_i/n$, $i=2,\dots,n$,

$$x_1=\frac{1}{n}\sum_{i=2}^n t_i+\frac{\delta n!}{n}\left(\prod_{i=2}^n t_i\right)^{-1}.$$

Clearly, this point lies on the surface (1.2) and, moreover, this point is also an element of H_δ. It is a convex combination of the points t_1e_1 and $t_i(e_1+e_i)$, $i=2,\dots,n$. We show that H_δ is actually a tangent hyperplane of (1.4). It is enough to show that the partial derivatives of (1.4) at that point coincide with the partial derivatives of H_δ:

$$\frac{\partial x_1}{\partial x_i}=1-\frac{\delta n!}{n^n}\frac{1}{x_i}\left(\prod_{j=2}^n x_j\right)^{-1}=1-\frac{t_1}{t_i}.$$

By (1.3) we get that on the set of all x such that

$$0\le x_i\le\frac{1}{n},\quad i=2,\dots,n\quad\text{and}\quad\frac{\delta n!}{n^{n-1}}\le\prod_{i=2}^n x_i$$

the surface (1.4) gives the boundary of $\bigcup H_\delta^-$. Therefore we get

$$\mathrm{vol}_n\left(S\cap\left(\bigcup H_\delta^-\right)\right)\ge\frac{\delta n!}{n^n}\int_{\delta n!/n}^{1/n}\cdots\int_{\gamma_k}^{1/n}\cdots\int_{\gamma_2}^{1/n}\left(\prod_{i=2}^n x_i\right)^{-1}dx_2\cdots dx_n$$

where

$$\gamma_k=\frac{\delta n!}{n^{n-k+1}}\left(\prod_{i=k+1}^n x_i\right)^{-1},\qquad k=2,\dots,n-1.$$

Since we have that

$$\frac{1}{(k-2)!}\int_{\gamma_k}^{1/n}\left(\prod_{i=k}^{n}x_i\right)^{-1}\left(\ln\left(\frac{n^{n-k+1}}{\delta n!}\prod_{i=k}^{n}x_i\right)\right)^{k-2}dx_k$$
$$=\frac{1}{(k-1)!}\left(\prod_{i=k+1}^{n}x_i\right)^{-1}\left(\ln\left(\frac{n^{n-k}}{\delta n!}\prod_{i=k+1}^{n}x_i\right)\right)^{k-1}$$

we obtain

$$\mathrm{vol}_n(S\cap(\cup H_\delta^-))\geq\frac{\delta n!}{n^n}\frac{1}{(n-1)!}\left(\ln\frac{1}{\delta n!}\right)^{n-1}=\frac{\delta}{n^{n-1}}\left(\ln\frac{1}{\delta n!}\right)^{n-1}.$$

Now we obtain the estimate from above. We blow up the simplex S by the factor n. We get here as above

$$\delta=\mathrm{vol}_n(nS\cap H_\delta^-)=\frac{1}{n!}\prod_{i=1}^{n}t_i,\qquad 0\leq t_i\leq n,\quad i=1,\ldots,n$$

and $x_i=t_i/n$ for $0\leq x_i\leq 1$, $i=2,\ldots,n$, and $\delta n!/n^n\leq\prod_{i=2}^{n}x_i$.

By using the computation for the lower estimate we get

$$\mathrm{vol}_n\left(S\cap(\cup H_\delta^-)\cap\left\{x\,\middle|\,\frac{\delta n!}{n^n}\leq\prod_{i=2}^{n}x_i\right\}\right)$$
$$\leq\mathrm{vol}_n\left(nS\cap(\cup H_\delta^-)\cap\left\{x\,\middle|\,\frac{\delta n!}{n^n}\leq\prod_{i=2}^{n}x_i\right\}\cap\left\{x\,\middle|\,0\leq x_i\leq 1,\ i=2,\ldots,n\right\}\right)$$
$$=n^n\mathrm{vol}_n\left(S\cap(\cup H_{\delta/n^n}^-)\cap\left\{x\,\middle|\,\frac{\delta}{n^n}\frac{n!}{n^{n-1}}\leq\prod_{i=2}^{n}x_i\right\}\right.$$
$$\left.\cap\left\{x\,\middle|\,0\leq x_i\leq\frac{1}{n},\ i=2,\ldots,n\right\}\right)$$
$$=n^n\frac{\delta}{n^n}\frac{1}{n^{n-1}}\left(\ln\frac{n^n}{\delta n!}\right)^{n-1}$$
$$=\frac{\delta}{n^{n-1}}\left(\ln\frac{n^n}{\delta n!}\right)^{n-1}.$$

Now it is left to estimate the volume of the part where $\prod_{i=2}^{n}x_i\leq\delta n!/n^n$ and $0\leq x_i\leq 1$, $i=2,\ldots,n$. We split this set again up into one set where at least one of the x_i, $i=2,\ldots,n$ are less than δ and therefore that volume is less than $(n-1)\delta$, and a second set where all x_i are larger than δ. Clearly the volume of the latter set is less than

$$\int_\delta^1 \cdots \int_\delta^1 \int_0^{(\delta n!/n^n)(\Pi_3^n x_i)^{-1}} dx_2 \cdots dx_n = \int_\delta^1 \cdots \int_\delta^1 \frac{\delta n!}{n^n}\left(\prod_3^n x_i\right)^{-1} dx_3 \cdots dx_n$$
$$= \frac{\delta n!}{n^n}\left(\ln\left(\frac{1}{\delta}\right)\right)^{n-2}.$$

We are proving (ii) and (iii) together. We use an inductive argument. The statement (ii) is easily checked in $\mathbf{R}^2$. Together with (i) this also proves (iii) in the case of $\mathbf{R}^2$. Assume now that we have verified (ii) and (iii) for the dimension $n-1$. We may assume that the simplex has equal sides and that $z_0 = -e_1$ and $z_1 = e_1$. Therefore the points $z_2, \ldots, z_n$ are contained in the subspace E orthogonal to e_1 and $E \cap S$ is the simplex spanned by $0, z_2, \ldots, z_n$. Consider now the convex floating body $(E \cap S)_\epsilon$ of $E \cap S$ with respect to the subspace E. We put

$$(E \cap S)_\epsilon \times [-e_1, e_1]$$

and observe that any hyperplane H so that $\mathring{H}^-$ contains z_0, z_1 and so that H touches $(E \cap S)_\epsilon \times [-e_1, e_1]$ cuts off a set of volume larger than $4^{-n}\epsilon$ provided that $2\epsilon \le \mathrm{vol}_{n-1}(E \cap S)$.

Now we conclude that

$$((E \cap S) \setminus (E \cap S)_\epsilon) \times [-e_1, e_1] \supseteq S \cap \left(\bigcup H_\delta^-\right)$$

where $\delta = 4^{-n}\epsilon$ and $\mathring{H}_\delta^-$ contains at least z_0 and z_1. Now we apply the induction hypothesis and obtain (ii) for the dimension n. (i) and (ii) for the dimension n give (iii) for the dimension n. ■

Lemma 1.4 *Let S be the simplex in $\mathbf{R}^n$ spanned by $z_1, \ldots, z_{n+1}$. Assume that S has a nonempty interior and let H_1 and H_2 be hyperplanes so that*

$$z_1, \ldots, z_{n-1}, \in H_1, H_2, \quad z_n \in \mathring{H}_1^-, \quad z_{n+1} \in \mathring{H}_2^-.$$

Let H_δ be hyperplanes so that $z_1 \in H_\delta^-$ and $z_2, \ldots, z_{n+1} \in H_\delta^+$ so that $\mathrm{vol}_n(S \cap H_\delta^-) = \delta$. Then there is a constant C so that we have for $\delta \le \frac{1}{2}\mathrm{vol}_n(S)$

$$\mathrm{vol}_n(S \cap (\cup H_\delta^-) \cap H_1^+ \cap H_2^+) \le C \frac{n!}{n^{n-1}}\, \delta \left(\ln \frac{n^{n-1}\mathrm{vol}_n(S)}{\delta}\right)^{n-2}$$

where C only depends on H_1 and H_2 and the volume of S.

Proof. We proceed here as in the proof of Lemma 1.3. We may assume that $z_1 = 0$, $z_2 = e_1$, $z_3 = e_1 + e_2, \ldots, z_{n+1} = e_1 + e_n$. For properly chosen a_1 and a_2 we have $H_1 = \{t \in \mathbf{R}^n \mid t_{n-1} = a_1 t_n\}$ and $H_2 = \{t \in \mathbf{R}^n \mid t_{n-1} = a_2 t_n\}$. Therefore we get as limits of integration for the variable t_{n-1} that $a_1 t_n \le t_{n-1} \le a_2 t_n$.

By the same reasoning as in the proof of Lemma 1.3 the boundary of $\cup H_\delta^-$ is below

$$x_1 = \sum_{i=2}^{n} x_i + \frac{\delta n!}{n^n}\left(\prod_{i=2}^{n} x_i\right)^{-1}$$

on the set under consideration.

The area where at least one of the x_i, $i = 2, \ldots, n$ is less than $\delta n!/n^n$ has $n-1$ dimensional volume less than $(n-1)\delta n!/n^n$. Therefore we get

$$\begin{aligned}
&\mathrm{vol}_n(S \cap (\cup H_\delta^-) \cap H_1^+ \cap H_2^+) \\
&\le \frac{\delta n!}{n^n} \int_{\delta n!/n^n}^{1} \int_{a_1 x_n}^{a_2 x_n} \int_{\delta n!/n^n}^{1} \cdots \int_{\delta n!/n^n}^{1} \left(\prod_{i=2}^{n} x_i\right)^{-1} dx_2 \cdots dx_n + \frac{\delta n!}{n^{n-1}} \\
&= \frac{\delta n!}{n^n}\left(\ln \frac{a_2}{a_1}\right)\left(\ln \frac{n^n}{\delta n!}\right)^{n-2} + \frac{\delta n!}{n^{n-1}} \le C \frac{n!}{n^{n-1}}\, \delta \left(\ln \frac{n^n}{\delta n!}\right)^{n-2}. \qquad \blacksquare
\end{aligned}$$

LEMMA 1.5. *Let S be the simplex in $\mathbf{R}^n$ spanned by $x_1, \ldots, x_{n+1}$. Assume that S has a nonempty interior and let $H_1, \ldots, H_n$ be hyperplanes such that*

$$(1.5) \quad x_1, \ldots, x_{k-1} \in H_k; \quad x_k \in \mathring{H}_k^+; \quad x_{k+1}, \ldots, x_{n+1} \in \mathring{H}_k^-, \quad k = 1, 2, \ldots, n.$$

(i) *Let H_δ be hyperplanes so that $x_1 \in H_\delta^-$ and $x_2, \ldots, x_{n+1} \in H_\delta^+$ so that $\mathrm{vol}_n(S \cap H_\delta^-) = \delta$. Then we have for sufficiently small $\delta > 0$*

$$\begin{aligned}
&\frac{1}{n!}\frac{1}{n^{n-1}}\,\delta\left(\ln \frac{n^n \mathrm{vol}_n(S)}{\delta}\right)^{n-1} - C\delta\left(\ln \frac{\mathrm{vol}_n(S)}{\delta}\right)^{n-2} \\
&\le \mathrm{vol}_n\left(S \cap (\cup H_\delta^-) \cap \left(\bigcap_{i=2}^{n} H_i^+\right)\right).
\end{aligned}$$

(ii) *Let H_δ be hyperplanes so that $\mathrm{vol}_n(S \cap H_\delta^-) = \delta$; then we have*

$$\begin{aligned}
&\mathrm{vol}_n\left(S \cap (\cup H_\delta^-) \cap \left(\bigcap_{i=1}^{n} H_i^+\right)\right) \\
&\le \frac{1}{n!}\frac{1}{n^{n-1}}\,\delta\left(\ln \frac{n^n \mathrm{vol}_n(S)}{\delta}\right)^{n-1} + C\delta\left(\ln \frac{\mathrm{vol}_n(S)}{\delta}\right)^{n-2}
\end{aligned}$$

where C depends on the hyperplanes H_i, $i = 1, \ldots, n$ and S.

PROOF. We may assume that $\|x_i - x_j\|_2 = 1$ if $i \neq j$ and $i, j = 1, \ldots, n+1$. There are $\binom{n}{2}$ hyperplanes H_{x_i,x_j}, $i \neq j$, $i,j = 2, \ldots, n+1$, so that $x_k \in H_{x_i,x_j}$ if $k \neq i,j$ and $\frac{1}{2}(x_i + x_j) \in H_{x_i,x_j}$. By this S is split up into $n!$ isometric pieces S_i, $i = 1, \ldots, n!$. Each of these pieces contains x_1. Therefore we get by Lemma 1.3(i)

$$\mathrm{vol}_n(S_i \cap (\cup H_\delta^-)) \geq \frac{1}{n!} \frac{1}{n^{n-1}} \delta \left(\ln \frac{\mathrm{vol}_n(S)}{\delta}\right)^{n-1}.$$

Now we are choosing hyperplanes K_{x_i,x_j}, $i \neq j$, $i,j = 2, \ldots, n+1$ such that $x_k \in K_{x_i,x_j}$ if $k \neq i,j$ and

$$S \cap \left(\bigcap_{i,j=2}^{n+1} K_{x_i,x_j}^-\right) \subseteq S \cap \left(\bigcap_{i=2}^{n} H_i^+\right).$$

It is easily seen that this in possible. Moreover, we may assume that $S \cap (\bigcap_{i,j=2}^{n+1} K_{x_i,x_j}^-) \subseteq S_1$ and that $S_1 = S \cap (\bigcap_{i,j=2}^{n+1} H_{x_i,x_j}^+)$. Now we subtract the sets $H_{x_i,x_j}^+ \cap K_{x_i,x_j}^+$. By Lemma 1.4 we have

$$\begin{aligned}
\mathrm{vol}_n\left(S \cap \left(\bigcap_{i=2}^{n} H_i^+\right) \cap (\cup H_\delta^-)\right) &\geq \mathrm{vol}_n\left(S_1 \cap \left(\bigcap_{i,j=2}^{n+1} K_{x_i,x_j}^-\right) \cap (\cup H_\delta^-)\right) \\
&\geq \mathrm{vol}_n(S_1 \cap (\cup H_\delta^-)) \\
&\quad - \sum_{i,j=2}^{n+1} \mathrm{vol}_n(S \cap K_{x_i,x_j}^+ \cap H_{x_i,x_j}^+ \cap (\cup H_\delta^-)) \\
&\geq \frac{1}{n!} \frac{1}{n^{n-1}} \delta \left(\ln \frac{\mathrm{vol}_n(S)}{\delta}\right)^{n-1} \\
&\quad - C\delta \left(\ln \left(\frac{\mathrm{vol}_n(S)}{\delta}\right)\right)^{n-2}.
\end{aligned}$$

(ii) is shown in the same way as (i). We have to use Lemmas 1.3 and 1.4. ∎

LEMMA 1.6. *Let P be a convex polytope with nonempty interior in $\mathbf{R}^n$. There is a family of simplices S_i, T_i, $i = 1, \ldots, \phi_n(P)$ and hyperplanes H_x, $x \in \mathrm{ext}(P)$, such that*

(i) $P \cap H_x^- \cap H_z^- = \varnothing$ *if* $x \neq z$.

(ii) $\mathring{S}_i \cap \mathring{S}_j = \varnothing$ *if* $i \neq j$.

(iii) *For every i there is $x \in \mathrm{ext}(P)$ so that $S_i \subseteq P \cap H_x^-$ and $P \subseteq T_i$.*

(iv) *For every* T_i, $i = 1, \ldots, \phi_n(P)$ *there are hyperplanes* H_j^i, $j = 1, \ldots, n$ *satisfying* (1.5) *of Lemma* 1.5 *such that*

$$T_i \cap \left(\bigcap_{j=1}^{n} H_j^{i+} \right) = S_i.$$

Proof. We construct the simplices by induction. For $n = 1$, P is an interval and the statement obvious. For $n \geq 2$ we choose for every extreme point a hyperplane H_x so that $x \in \mathring{H}_x^-$ and

$$P \cap H_x^- \cap H_z^- = \emptyset$$

for $x \neq z$.

$P \cap H_x$ has simplices $\tilde{S}_i$, $\tilde{T}_i$, $i = 1, \ldots, \phi_{n-1}(P \cap H_x)$ satisfying (i), (ii), (iii), and (iv) for $P \cap H_x$. Moreover, we have for $P \cap H_x$ hyperplanes $\tilde{H}_j^i$, $j = 1, \ldots, n-1$ satisfying (iv). We define

$$S_i = [x, \tilde{S}_i]$$

and H_{j+1}^i is the hyperplane spanned by x and $\tilde{H}_j^i$. For H_1^i we choose H_x. Now we construct T_i. The extreme point x and $\tilde{T}_i$ generate a cone that contains P. To get T_i we intersect this cone with a halfspace that contains P and whose defining hyperplane is parallel to H_x. In this way we obtain for every x a sequence of simplices. By the definition of $\phi_n(P)$ it follows that we get in fact $\phi_n(P)$ simplices.

■

Lemma 1.7. *Let P be a convex polytope with nonempty interior in* $\mathbf{R}^n$. *Then there are simplices* S_i, T_i, $i = 1, \ldots, \psi_n(P)$ *and hyperplanes* H_j^i, $j = 1, \ldots, n+1$ *so that*

(i) $\mathring{S}_i \cap \mathring{S}_j = \emptyset$ *for* $i \neq j$,

(ii) $\bigcup_{i=1}^{\psi_n(P)} S_i = P$,

(iii) $S_i \subseteq T_i \subseteq P$, $i = 1, \ldots, \psi_n(P)$,

(iv) $\bigcap_{j=1}^{n+1} H_j^{i+} = S_i$, $i = 1, \ldots, \psi_n(P)$,

(v) H_j^i, $j = 1, \ldots, n$ *satisfy the hypothesis* (1.5) *of Lemma* 1.5 *with respect to* T_i.

Proof. We proceed by induction over the dimension n. For $n = 1$, P is an interval and the choices are obvious. For $n \geq 2$ we choose an interior point $x \in P$. Each $n-1$ dimensional face F of P has simplices $\tilde{S}_i$, $\tilde{T}_i$, $i = 1, \ldots, \psi_{n-1}(F)$ and hyperplanes $\tilde{H}_j^i$, $j = 1, \ldots, n-1$ satisfying (i)–(v). Again, we define

$$S_i = [x, \tilde{S}_i].$$

The hyperplanes H_j^i are spanned by x and $\tilde{H}_j^i$, $j = 1, \ldots, n$. As H_{n+1}^i we choose the hyperplane containing F. Now we choose an interior point $\tilde{x}$ of P so that x is an interior point of

$$T_i = [\tilde{x}, \tilde{T}_i]$$

and so that the center of gravity of S_i is on the line through x and $\tilde{x}$. ■

Proof of Theorem 1.2. Let S_i, $i = 1, \ldots, \phi_n(P)$ be as given in Lemma 1.6 and H_δ hyperplanes so that $\mathrm{vol}_n(P \cap H_\delta^-) = \delta$. Let K_δ^i be hyperplanes so that $\mathrm{vol}_n(T_i \cap K_\delta^{i-}) = \delta$ and K_δ^{i-} contains the only extreme point of T_i that is also an extreme point of P. Since $P \subseteq T_i$ we have for all $i = 1, \ldots, \phi_n(P)$

$$\bigcup H_\delta^- \supseteq \bigcup K_\delta^{i-}.$$

This implies

$$\mathrm{vol}_n(P \cap (\cup H_\delta^-)) \geq \sum_{i=1}^{\phi_n(P)} \mathrm{vol}_n\left(S_i \cap \left(\bigcup K_\delta^{i-}\right)\right).$$

Now we apply Lemma 1.5(i) and get

$$\mathrm{vol}_n\left(P \cap \left(\bigcup H_\delta^-\right)\right) \geq \frac{1}{n!}\,\frac{1}{n^{n-1}}\,\phi_n(P)\delta\left(\ln \frac{n^n \mathrm{vol}_n(S)}{\delta}\right)^{n-1} - C\phi_n(P)\delta\left(\ln \frac{\mathrm{vol}_n(S)}{\delta}\right)^{n-2}.$$

The opposite inequality is obtained in the same way by using Lemmas 1.5(ii) and 1.7. ■

Lemma 1.8. *Let P be a convex polytope in $\mathbf{R}^n$ with nonempty interior. Let H_δ denote hyperplanes so that* $\mathrm{vol}_n(P \cap H_\delta^-) = \delta$. *Then we have for all δ, $0 < \delta < \delta_n$,*

$$\mathrm{vol}_n(P \backslash P_\delta) \leq \frac{\phi_n(P)}{n!\,n^{n-1}}\,\delta\left(\ln \frac{\mathrm{vol}_n(P)}{\delta}\right)^{n-1} + c_n\phi_n(P)\delta\left(\ln \frac{\mathrm{vol}_n(P)}{\delta}\right)^{n-2},$$

where c_n and δ_n depend only on the dimension n.

Lemma 1.9 [J]. *Let K be a convex body in $\mathbf{R}^n$. Then there is an ellipsoid $\mathcal{E}(z,r)$ with center z and radius $r = (r_1, \ldots, r_n)$,*

$$\mathcal{E}(z,r) \subseteq K \subseteq \mathcal{E}(z, n \cdot r). \tag{1.6}$$

Proof of Lemma 1.8. We use a refinement of the construction of Lemma 1.7. The refinement consists of the specification how x and $\tilde{x}$ in the proof of Lemma 1.7 are chosen.

By Lemma 1.9 for any convex body in $\mathbf{R}^n$ there is an ellipsoid $\mathcal{E}(z,r)$ satisfying (1.6). As x we choose the center z of the ellipsoid $\mathcal{E}(z,r)$. Next we choose

$$y \text{ as the center of gravity of } \tilde{S}_i. \tag{1.7}$$

The line through y and x has two intersections with ∂K, namely y and another point y'. As $\tilde{x}$ we choose y'. Because of (1.6) we have

$$\|x-y\|_2 \le n\|x-\tilde{x}\|_2. \tag{1.8}$$

By an affine transform we map S_i onto $[0, e_1, e_2, \ldots, e_n]$. Because of (1.7) and (1.8) we get that the image of T_i under the same affine transform contains the simplex spanned by

$$0 \quad \text{and} \quad -\frac{\sqrt{k-1}}{k^2}(e_1+\cdots+e_{k-1})+\frac{k+1}{k}e_k, \qquad k=1,\ldots,n.$$

Therefore, if we apply Lemma 1.5 now to this construction, the constants will only depend on the dimension n. ■

2. Estimates from below for polyhedral approximation

The symmetric difference $d_s(K_1,K_2)$ between two convex bodies K_1, K_2 is $\mathrm{vol}_n(K_1 \Delta K_2)$.

Proposition 2.1. *Let K be a convex body in $\mathbf{R}^n$. Then we have for all polytopes P, $P \subseteq K$, and all δ, $0<\delta<\delta_n$,*

$$\frac{1}{2}\mathrm{vol}_n(K\backslash K_\delta) \le d_s(K,P)$$

provided that

$$\mathrm{vol}_n(K\backslash K_\delta) \ge 2\frac{\phi_n(P)}{n!n^{n-1}}\delta\left(\ln\frac{\mathrm{vol}_n(P)}{\delta}\right)^{n-1}+c_n\phi_n(P)\delta\left(\ln\frac{\mathrm{vol}_n(P)}{\delta}\right)^{n-2}$$

where c_n is a constant that depends only on n.

Proof. Since $P \subseteq K$ we have $P_\delta \subseteq K_\delta$. Therefore we get

$$\mathrm{vol}_n(K\backslash P) \ge \mathrm{vol}_n(K\backslash K_\delta)-\mathrm{vol}_n(P\backslash K_\delta) \ge \mathrm{vol}_n(K\backslash K_\delta)-\mathrm{vol}_n(P\backslash P_\delta).$$

Now we apply Lemma 1.8. ■

Corollary 2.2. *Let K be a convex body in $\mathbf{R}^n$ and P, $P \subseteq K$, a polytope with nonempty interior. Then we have for sufficiently large $\phi_n(P)$*

$$d_s(K,P) \geq Dn^5 \phi_n(P)^{-2/(n-1)} \left(\int_{\partial K} \kappa(x)^{1/(n+1)} d\mu(x) \right)^{(n+1)/(n-1)}$$
$$\times \left(\ln \frac{\phi_n(P) \mathrm{vol}_n(P)^{(n-1)/(n+1)}}{\int_{\partial K} \kappa(x)^{1/(n+1)} d\mu(x)} \right)^{-2},$$

Proof. By Theorem 1 in [Schü-W] we have

$$\lim_{\delta \to 0} \frac{\mathrm{vol}_n(K \backslash K_\delta)}{\delta^{2/(n+1)}} = \frac{1}{c_n} \int_{\partial K} \kappa(x)^{1/(n+1)} d\mu(x) \tag{2.1}$$

with

$$c_n = 2\left(\frac{1}{n+1} \mathrm{vol}_{n-1}(B_2^{n-1}) \right)^{2/(n+1)}.$$

The hypothesis of Proposition 2.1 is fulfilled if

$$\delta \leq \frac{1}{2} t \left(\ln \frac{\mathrm{vol}_n(P)}{t} \right)^{-n-1} \tag{2.2}$$

with

$$t = \left\{ \frac{1}{8} \frac{n! n^{n-1}}{c_n \phi_n(P)} \int_{\partial K} \kappa(x)^{1/(n+1)} d\mu(x) \right\}^{(n+1)/(n-1)}$$

and t is sufficiently small, i.e. $\phi_n(P)$ is sufficiently large. Now we use $d_s(K,P) \geq \frac{1}{2}\mathrm{vol}(K \backslash K_\delta)$, (2.1), and (2.2). ■

Corollary 2.3. *Let K be a convex body in $\mathbf{R}^n$, and $P, P \subseteq K$, a simplicial polytope. Then we have for sufficiently large $\#\mathrm{fac}_{n-1}(P)$*

$$d_s(K,P) \geq Dn^3 (\#\mathrm{fac}_{n-1}(P))^{-2/(n-1)} \left(\int_{\partial K} \kappa(x)^{1/(n+1)} d\mu(x) \right)^{(n+1)/(n-1)}$$
$$\times \left(\ln \frac{\#\mathrm{fac}_{n-1}(P) \mathrm{vol}_n(P)^{(n-1)/(n+1)}}{\int_{\partial K} \kappa(x)^{1/(n+1)} d\mu(x)} \right)^{-2}.$$

To prove Corollary 2.3 we use Corollary 2.2 and $\psi_n(P) = n! \#\mathrm{fac}_{n-1}(P)$. ■

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